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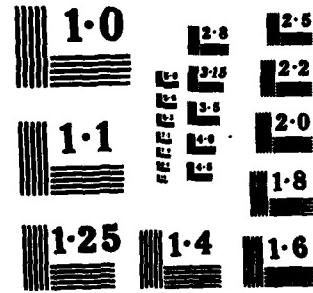
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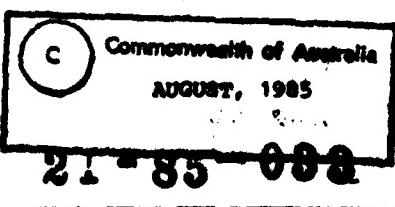
E.H. van Leeuwen

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E.H. van Leeuwen

ABSTRACT

In this paper the time evolution of weak shock waves propagating within a fluid-gas mixture is considered. The model uses continuum classical theory to describe the shock waves and allows for the relative motion of the gas bubbles and liquid. It is shown that for a dissipative system the waves are governed by Burger's Equation. This corresponds to the far-field solution for waves of small amplitude and long wavelength. The model also provides a non-linear description of how the compression wave forms and approaches a steady state.

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Burger's equation and shock waves propagating within
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BURGER'S EQUATION AND SHOCK WAVES
PROPAGATING WITHIN LIQUID-GAS MIXTURES

1. INTRODUCTION

The transient behaviour occurring when a weak shock wave propagates into an inviscid and incompressible liquid containing small gas bubbles is examined in the long-time regime. A semi-infinite shock-tube filled with a liquid-gas mixture and fitted at one end with a moveable piston is considered. When the piston is set into motion at constant velocity (u_p), a wave propagates into the liquid-gas mixture and is modified by dissipative processes and by non-linear effects associated with convection. The dissipative processes involved in this analysis are viscosity and thermal conductivity. The effects produced by these processes upon the shock-wave propagation can be complicated since both waveform and velocity are modified under their influence. Non-linear effects also modify the wave form and become important when of comparable magnitude to the dissipative processes. This situation will arise if viscosity and thermal conductivity cause velocity gradients to decrease. Thus in the far field, we could expect the exact solution to approximate a steady compression wave. Since viscous forces exert their greatest effect in the region of the shock transition, this region will be discussed in detail.

The linear theory (van Leeuwen (1984)) starts with the Navier-Stokes equations for a fluid gas mixture and expands the dependent variables using a perturbation series. By retaining terms to order $O(\epsilon)$, the equations of mass, momentum and energy can be solved for both the near field and far field solutions. In the case of the far field, the solutions are given in Appendix 1. These solutions can be shown to break down, when t becomes large, and this feature can be illustrated as follows. The width of the shock transition zone (Thompson (1972), Gilbarg and Paolucci (1953)) is:

$$\delta = \frac{u_p}{\left(\frac{\partial u}{\partial x} \right)_{\max}},$$

for the case where the velocity, as given by A1, is of order $O(\sqrt{t})$. Hence as t increases the size of the shock transition zone also increases. It is well known however that δ is of order $O(\epsilon^{-1})$ for steady weak shock waves, where ϵ is an appropriate perturbation parameter; thus when t becomes large compared to ϵ^{-2} , the linearized solution yields an excessively broad shock wave and so fails to describe the phenomena adequately.

In the present paper a uniformly satisfactory far-field solution is sought for a shock wave propagating into a liquid-gas mixture. Using "stretched" coordinates and singular perturbation methods a uniformly valid solution for $t + \infty$ can be derived. Conditions at infinity and elsewhere are assumed uniform.

An analogous type of piston problem has been studied and solved by Moran and Shen (1965) (Referred subsequently to as MS), for a viscous heat-conducting gas, using continuum theory. They found that the linearized Navier-Stokes equations are valid for times as large as $O(\epsilon^{-2})$ mean free time after the piston is set in motion, while at larger times the solution is governed by Burger's equation. Crespo (1969) has considered the propagation of infinitesimally small sound waves in a liquid containing gas bubbles, taking into account the relative motion of the bubbles and liquid. Also the structures of shock waves in liquids containing gas bubbles have been examined by Campbell and Pitcher (1958) and van Wijngaarden (1970). The methodologies of MS and Crespo are closely followed in this paper. A continuum description of the liquid-gas mixture is given in section 2 together with simplifying assumptions. The method of analysis employed is to construct the exact form of the Navier-Stokes (NS) equations for the conservation of mass, energy and momentum in a one-dimensional viscous unsteady flow, together with boundary conditions appropriate to the piston problem. The dependent variables appearing in the NS equations are then expanded in the usual form:

$$\psi = \psi^{(0)} + \epsilon \psi^{(1)} + \epsilon^2 \psi^{(2)} + \dots,$$

and terms of order $O(\epsilon^2)$ are retained. The NS equations to order $O(\epsilon)$ adopt a steady-state form in the wave frame, while to order $O(\epsilon^2)$ the time evolution effects come into play.

2. BASIC EQUATIONS AND MODEL

The liquid host-medium is assumed to be impregnated with gas bubbles of fixed size and homogeneous distribution. The liquid is also assumed incompressible and inviscid. (The gas obeys the perfect gas law.) The state of the liquid can be described in terms of its velocity u_l , pressure P_l , density ρ_l and temperature T_l ; for the gas the corresponding quantities are u_g , P_g , ρ_g and T_g . The volume fraction of the bubbles in an aerated liquid with initial volume V_0 is denoted by β , so that $(1-\beta)$ is the volume fraction of the fluid. The liquid is taken to have a large heat capacity and that

variations in pressure and temperature at the equilibrium state can be neglected, i.e.,

$$\left. \begin{array}{l} T_l = T_g = 0 \\ P_l = P_g = 0 \end{array} \right\} \text{at equilibrium}$$

The above condition with respect to pressure implies that the radius of the gas bubbles is small compared to the width of the shock transition zone. Further assumptions made are (i) that the viscous forces are attributable to gas bubble drag, (ii) that heat transfer occurs only between the gas bubbles and surrounding liquid, and (iii) that the thermal motion of the gas bubbles is negligible and so does not contribute to the pressure.

Both dependent and independent variables are initially expressed in a coordinate system designated in terms of " $\hat{\psi}$ " (i.e., $\hat{\psi} = (u, P, \rho, \dot{S}, T)$), but for convenience the $\hat{\cdot}$ is dropped from all equations.

In the nomenclature described above, the exact Navier-Stokes equations (see Ockendon and Tayler (1983)) expressing the conservation of mass, momentum and energy of a liquid-gas system become for a one-dimensional unsteady flow (van Leeuwen (1984), Crespo (1969), MS) as:

(i) Equation of continuity:

$$\partial_t(1-\beta) + \partial_x(1-\beta)u_l = 0, \quad (2.1a)$$

(ii) Equation of continuity for the gas:

$$\partial_t(\beta\rho_g) + \partial_x(\beta\rho_g u_g) = 0, \quad (2.1b)$$

(iii) Equation of momentum for the liquid-gas composition:

$$\rho_l(1-\beta)(\partial_t u_l + u_l \partial_x u_l) + \partial_x P_l = 0, \quad (2.1c)$$

(iv) Equation of momentum for the gas:

$$\partial_x P_l - \frac{\Gamma}{2} \rho_l (\partial_t u_{lg} + u_g \partial_x u_{lg}) - \frac{9}{2} \rho_l v_l \left(\frac{4\pi \rho_g}{3m} \right)^{2/3} u_{lg} = 0, \quad (2.1d)$$

where $u_{lg} = u_l - u_g$.

(v) Equation of energy for the gas:

$$\partial_t p_l + u_g \partial_x p_l - \frac{\gamma_g p_l}{\rho_g} (\partial_t \rho_g + u_g \partial_x \rho_g) - \\ - 3(\gamma_g - 1)\sigma_l \left(\frac{4\pi \rho_g}{3m} \right)^{2/3} (T_l - T_g) = 0, \quad (2.1e)$$

where ∂ denotes partial differentiation (elsewhere the alternative notation ' $,$ ' will also be used).

(vi) The equation of state for a thermally-perfect single-component, dilute gas is given by

$$p_g = \rho_g R T_g, \quad (2.1f)$$

where R is the universal gas constant. The corresponding liquid equation of state is simply $\rho_l = \text{constant}$.

In the above equations ν_l and σ_l are the kinematic viscosity, and the thermal conductivity of the fluid respectively, γ_g is the ratio of specific heats of the gas, and Γ is a function which depends on β and which for β small is essentially unity (Crespo (1969)).

One approach to solving the above equations is to apply the fundamental principle of dimensional analysis (ie., every problem must be expressible in terms of dimensionless variables). Firstly we seek dimensionless variables for the independent variables (x, t) . An examination of the physical constants appearing in (2.1) reveals shows that a constant with dimensions of length, can be formed which renders x and t dimensionless. That is, we can choose without loss of generality a length scale $k = a_e r_e u_p^{-1}$, where r_e is the radius of a gas bubble and $a_e = \sqrt{RT_e}$ is the equilibrium speed of sound in the liquid, such that:

$$\hat{x} = kx, \quad (2.2a)$$

$$\hat{t} = \frac{k}{a_e} t. \quad (2.2b)$$

Similarly the dependent variables $\hat{\psi} = (\hat{u}, \hat{p}, \hat{\rho}, \hat{\beta}, \hat{T})$ can also be reduced to dimensionless form, following the example of MS.

The dimensionless fluid variables can be expressed asymptotically in terms of fluid variables as a power series in ϵ about the equilibrium state $\{0, P_e, \rho_e, \beta_e, T_e\}$ ie.,

$$\hat{\psi} = \psi_e (1 + \epsilon \hat{\psi}^{(1)} + \epsilon^2 \hat{\psi}^{(2)} + \dots), \quad (2.3a)$$

for the state variables $\hat{P}, \hat{\rho}, \hat{\beta}$ and \hat{T} , and as

$$\hat{\psi} = \epsilon a_e (\psi^{(1)} + \epsilon \psi^{(2)} + \dots), \quad (2.3b)$$

for \hat{u} . The dimensionless perturbation variable ϵ is chosen as a measure of the piston Mach number

$$\epsilon = \frac{u_p}{a_e}, \quad (2.4)$$

where u_p is the initial piston velocity.

The linear theory fails when the diffusion thickness and the compression wave thickness are equal, i.e., when $t = \epsilon^{-2}$. To find a uniformly valid solution for $t > \epsilon^{-2}$ (i.e., for the far field) where the linear theory is no longer valid, it is necessary to introduce "stretched" coordinates for the dimensionless independent variables, in order to take into account slow variations in the wave form. This can be achieved by introducing the following scale transformation for the independent variables (Su and Gardiner (1969)):

$$\xi = \epsilon^\eta (x - \lambda t), \quad (2.5a)$$

$$\tau = \epsilon^{\eta+1} t, \quad (2.5b)$$

where λ is a dimensionless constant related to the velocity of propagation of the disturbance, and is to be determined by the analysis. The quantity η is a positive constant (also to be determined) such that time variations of any state variable are balanced by both non-linear and dissipative effects.

In the present nomenclature, the system of equations (2.1) can be expressed in dimensionless variables. The scheme is illustrated by considering the continuity equation (2.1a). After a change of independent variables from (x, t) to (ξ, τ) equation (2.1a) becomes:

$$a_e \partial_t (1-\beta) + \partial_x (1-\beta) u_\xi = 0. \quad (2.6)$$

Employing (2.5a) and (2.3) we obtain to order $O(\epsilon)$ the steady state equation:

$$\lambda \beta_e \partial_\xi \beta^{(1)} + (1-\beta_e) \partial_\xi u_\xi^{(1)} = 0, \quad (2.7a)$$

and to order $O(\epsilon^2)$:

$$\lambda \beta_e \partial_\xi \beta^{(2)} - \beta_e \partial_\tau \beta^{(1)} + (1-\beta_e) \partial_\xi u_\xi^{(2)} - \beta_e \beta^{(1)} \partial_\xi u_\xi^{(1)}$$

$$-\beta_e u_\xi^{(1)} \partial_\xi \beta^{(1)} = 0, \quad (2.10a)$$

which now incorporates time-evolution effects. Proceeding in a like manner for the remaining equations we obtain to order $O(\epsilon)$:

$$T_g^{(1)} = u_g^{(1)} - u_\xi^{(1)} = \rho_g^{(1)} - \rho_\xi^{(1)} = 0, \quad (2.7b)$$

$$\partial_\xi p_g^{(1)} - (1-\beta_e) \lambda \partial_\xi u_\xi^{(1)} = 0, \quad (2.7c)$$

$$\partial_\xi u_g^{(1)} - \lambda \partial_\xi (\rho_g^{(1)} + \beta^{(1)}) = 0. \quad (2.7d)$$

Integrating equation (2.7) and equating the constants of integration to zero gives the state variables of the gas in terms of the fluid velocity:

$$u_g^{(1)} = u_\xi^{(1)} \left((1, \lambda(1-\beta_e), \frac{1}{\lambda\beta_e}, -\frac{(1-\beta_e)}{\lambda\beta_e}, 0) \right), \quad (2.8)$$

where the dimensionless velocity λ satisfies the relation

$$\lambda^2 = \frac{1}{\beta_e(1-\beta_e)}, \quad (2.9)$$

together with the obvious physical constraint that the fluid-gas mixture is not purely fluid ($\beta_e = 0$) or gaseous ($\beta_e = 1$). This velocity has a minimum value when $\beta_e = 1/2$.

Following a similar procedure to the above for the equation of order $O(\epsilon^2)$, in which the effects of dissipation are embedded in the flow, we find that the parameter η should be chosen as unity. For this choice of η the second order perturbation equation, together with (2.7), can be expressed as:

$$\begin{aligned} -i\partial_\xi (\rho_g^{(2)} + \beta^{(2)} + p_g^{(1)} \beta^{(1)}) + \partial_\xi (u_g^{(2)} + p_g^{(1)} u_\xi^{(1)} + \beta^{(1)} u_\xi^{(1)}) \\ + \partial_\tau (p_g^{(1)} + \beta^{(1)}) = 0, \end{aligned} \quad (2.10b)$$

$$\begin{aligned} \lambda \beta_e \beta^{(1)} \partial_\xi u_\xi^{(1)} - \lambda(1-\beta_e) \partial_\xi u_\xi^{(2)} + (1-\beta_e) \partial_\tau u_\xi^{(1)} + (1-\beta_e) u_\xi^{(1)} \partial_\xi u_\xi^{(1)} \\ + \partial_\xi p_g^{(2)} = 0, \end{aligned} \quad (2.10c)$$

$$\partial_{\xi} p_g^{(1)} + \phi(u_{\xi}^{(2)} - u_g^{(2)}) = 0, \quad (2.10d)$$

$$\lambda(1-\gamma_g) \partial_{\xi} p_g^{(1)} - \frac{\gamma_g r_e}{u_p \tau_T} T_g^{(2)} = 0, \quad (2.10e)$$

$$p_g^{(2)} - \rho_g^{(2)} - T_g^{(2)} = 0, \quad (2.10f)$$

where $\phi = \frac{9}{2} \frac{k}{a_e} v_L (\frac{4\pi \rho_e}{3m})^{2/3}$ is a dimensionless constant which, in terms of the momentum relaxation coefficient $a_m = \frac{2}{9} \frac{r_e^2}{v_L}$, can be expressed as $\phi = \frac{k}{r_e a_m}$. The constant $\tau_T = \frac{mc_p}{4\pi \sigma_L r_e}$ is the thermal relaxation coefficient.

3. BURGER'S EQUATION FOR THE SHOCK WAVE STRUCTURE

First-order perturbations in the state variables appearing in (2.10) can be written in terms of $u_{\xi}^{(1)}$, via the steady state equations (2.7). Equations (2.10) can then be solved to give $\beta_{\xi}^{(2)}$ in terms of $u_{\xi}^{(2)}$ and $u_{\xi}^{(1)}$. The partial derivatives $\partial_{\xi} \beta_{\xi}^{(2)}$ are

$$\partial_{\xi} \beta_{\xi}^{(2)} = \frac{1}{\lambda \beta_e} (G_1 - (1-\beta_e) \partial_{\xi} u_{\xi}^{(2)}), \quad (3.1a)$$

$$\partial_{\xi} p_g^{(2)} = -G_2 + \lambda(1-\beta_e) \partial_{\xi} u_{\xi}^{(2)}, \quad (3.1b)$$

$$\partial_{\xi} \rho_g^{(2)} = \frac{1}{\lambda \beta_e} (\beta_e G_3 + \partial_{\xi} u_{\xi}^{(2)}), \quad (3.1c)$$

$$\partial_{\xi} u_g^{(2)} = \partial_{\xi} u_{\xi}^{(2)} - \frac{\lambda}{\phi} (1-\beta_e) \partial_{\xi} \beta_{\xi}^{(1)}, \quad (3.1d)$$

$$T_g^{(2)} = \frac{u_p \tau_T}{\gamma_g r_e \beta_e} (1 - \gamma_g) u_{\xi, \xi}^{(1)}, \quad (3.1e)$$

where G_j ($j=1,2,3$) are functions of the first order perturbation variables $\rho_g^{(1)}$, $u_{\xi}^{(1)}$ and $\beta_{\xi}^{(1)}$ ie.,

$$G_1 = \partial_\tau \beta^{(1)} + \partial_\xi (u_\xi^{(1)} \beta^{(1)}), \quad (3.2a)$$

$$G_2 = (1-\beta_e) \partial_\tau u_\xi^{(1)} + \partial_\xi (u_\xi^{(1)} (\frac{1}{2}(1-\beta_e) u_\xi^{(1)} + \lambda \beta_e)), \quad (3.2b)$$

$$G_3 = \frac{1}{\lambda} \partial_\xi (\frac{1}{\phi} \partial_\xi p_g^{(1)} + p_g^{(1)} (u_\xi^{(1)} - \gamma \beta^{(1)})). \quad (3.2c)$$

Differentiating equation (3.2e) with respect to ξ , and substituting for $T_g^{(2)}$ using (2.10f) together with (3.2), we find that $u_\xi^{(1)}$ satisfies Burger's equation (Whitham (1974)):

$$\beta_e \partial_\tau u_\xi^{(1)} + u_\xi^{(1)} \partial_\xi u_\xi^{(1)} - \frac{\theta}{2} \partial_{\xi\xi} u_\xi^{(1)} = 0 \text{ for } \tau > 0, \quad (3.3)$$

where $\theta = (\frac{\beta_e}{\phi} + \frac{u_p \tau_T (\gamma_g - 1)}{\gamma_g \tau_e (1 - \beta_e)})$ is a dimensionless constant which behaves as a

viscosity-like term and is typically the inverse of the Reynolds number. This term serves to prevent singularities which would occur if $\theta = 0$. In equation (3.3) $u_\xi u_\xi \xi$ is a non-linear convective term and $\frac{1}{2} \theta u_\xi \xi \xi$ is a diffusive equation viscous term.

The algebraic form of the above equation confirms that the stretched coordinates (2.5) are appropriate, since equation (3.3) retains terms describing time-dependent flows with non-linear convection and dissipation.

4. SOLUTIONS FOR A DISCONTINUOUS SHOCK

The general initial-value problem for Burger's equation (3.3) can be solved exactly by means of a Cole-Hopf transformation (Cole (1951), Hopf (1950)). This transformation reduces Burger's equation to a classical heat-conduction equation which can then be solved using integral-transform methods. In a coordinate system $(\bar{\xi}, \bar{\tau})$, where $\bar{\xi} = \xi$ and $\bar{\tau} = \beta_e^{-1} \tau$, we can introduce a function $\Sigma(\bar{\xi}, \bar{\tau})$, which is related to u_ξ , in such a way that:

$$\Sigma, \bar{\xi} = -\frac{1}{\theta} u_\xi^{(1)} \Sigma, \quad (4.1a)$$

$$\Sigma, \bar{\tau} = \frac{1}{\theta} (\frac{1}{2} u_\xi^{(1)} \bar{\tau}^2 - \frac{\theta}{2} u_\xi^{(1)} \Sigma, \bar{\xi}), \quad (4.1b)$$

and this may be achieved by noting that the integrability condition is precisely equation (3.3) (i.e., $\partial_{\bar{\tau}} (4.1a) - \partial_{\bar{\xi}} (4.1b) = 0$, yields equation (3.3)). This non-linear transformation eliminates the non-linear terms

appearing in (3.3). Integration of (4.1a) then yields:

$$\Sigma = \exp\left(-\frac{1}{\theta} \int u_{\xi}^{(1)} d\xi\right). \quad (4.2)$$

On substituting equation (4.2) into equation (3.3) we find that Σ is a solution of the classical heat conduction equation:

$$\Sigma_{,\bar{\tau}} = \frac{\theta}{2} \Sigma_{,\bar{\xi}\bar{\xi}}. \quad (4.3)$$

The initial-value problem for Burger's equation is therefore exactly soluble and pseudo-stationary solutions can be obtained from the ansatz:

$$\psi(\bar{\xi},\bar{\tau}) = \psi(\bar{\xi} - a\bar{\tau}),$$

from which it follows that

$$\psi(\bar{\xi}) = a(1 + \exp(-c(\bar{\xi} - \bar{\xi}_0))).$$

More explicitly, solutions of (4.3) corresponding to a discontinuous shock wave at $\bar{\xi} = 0$ can readily be found. For the case that the initial condition at $\bar{\tau} = 0$ is

$$u_{\xi}^{(1)}(\bar{\xi},0) = \begin{cases} u_0, & \bar{\xi} < 0 \\ 0, & \bar{\xi} > 0, \end{cases}$$

equation (3.2) yields:

$$\Sigma(\bar{\xi},0) = \begin{cases} \frac{u_0 \bar{\xi}}{\theta}, & \bar{\xi} < 0 \\ 1, & \bar{\xi} > 0, \end{cases}$$

where Σ is a continuous function through $\bar{\xi} = 0$ and u_0 is an arbitrary constant which can be set equal to unity without loss of generality. This initial condition is the one which matches the asymptotic form of the near-field solution (see appendix 1). The solution of (3.3) with this initial condition can be found by taking a Laplace transform of (3.3) with respect to $\bar{\tau}$. This yields, in untransformed coordinates (Ockendon and Tayler (1983)):

$$u_{\xi}^{(1)}(\bar{\xi},\bar{\tau}) = u_0 \left[1 + \frac{\exp\left(\frac{u_0}{\theta}(\bar{\xi} - \frac{u_0 \bar{\tau}}{2})\right) \operatorname{erfc}\left(\frac{-\bar{\xi}}{\sqrt{2\theta\bar{\tau}}}\right)}{\operatorname{erfc}\left(\frac{\bar{\xi} - u_0 \bar{\tau}}{\sqrt{2\theta\bar{\tau}}}\right)} \right]^{-1}, \quad (4.5a)$$

where $\text{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x e^{-a^2} da$. The initial condition can be recovered from (4.5a) if $\xi \neq 0$ and $t = 0$. In coordinates (x, t) the fluid velocity (4.5a) may be expressed as

$$u_{\xi}^{(1)}(x, t) = u_0 \left[1 + \frac{\frac{\epsilon u_0}{\theta} \left(x - t(\lambda + \frac{u_0 \epsilon}{2\beta_e}) \right) \text{erfc}(\frac{-(x-\lambda t)}{\sqrt{2\theta t/\beta_e}})}{\text{erfc}(\frac{x - t(\lambda + u_0 \epsilon / \beta_e)}{\sqrt{2\theta t/\beta_e}})} \right]^{-1}. \quad (4.5b)$$

This solution is similar to that of MS for shock waves propagating into a viscous heat-conducting gas in the far field. The MS solution is recovered if $\theta \rightarrow 2\beta(\gamma + 1)$, $\beta_e \rightarrow 2(\gamma + 1)^{-1}$ and $\lambda \rightarrow \sqrt{\gamma}$.

If we let $\bar{t} \rightarrow \infty$ with ξ kept finite, then (4.5b) reduces to

$$u_{\xi}^{(1)}(x, t) = u_0 \left(1 + \frac{\epsilon u_0}{\theta} \left(x - t(\lambda + \frac{u_0 \epsilon}{2\beta_e}) \right) \right)^{-1}, \quad (4.6)$$

where the following properties of error functions, $\text{erfc}(0) = 1$ and $\text{erfc}(-\infty) = 2$ have been used. This solution represents a shock wave of

constant profile propagating with a velocity of $\lambda + \frac{u_0 \epsilon}{2\beta_e}$. This velocity is a

second-order approximation to the Rankine-Hugoniot result for the velocity of a weak shock (cf Taylor (1910)). The first-order approximation is given by (2.9). We further observe that if the solution (4.6) is expressed as

$$g(x, t) = -\frac{u_0}{2} + u_{\xi}^{(1)}(x, t)$$

then $g(x, t)$ satisfies the steady-state Burger Equation (cf MS where the asymptotic solution is missing a factor of $\frac{1}{2}$) ie.,

$$g g_x - \frac{\theta}{2} g_{xx} = 0.$$

5. NUMERICAL ILLUSTRATION

The foregoing solution (4.5b) is illustrated for a shock wave propagating into water ($\rho_1 = 10^3 \text{ kgm}^{-3}$, $a_1 = 1.51 \times 10^3 \text{ ms}^{-1}$, $v_1 = 8.57 \times 10^{-2} \text{ m s}^{-1}$, $\sigma_1 = 6.1 \times 10^{-44} \text{ J/(ms}^2\text{K)}$, $T_1 = 300^\circ\text{K}$) containing gas bubbles of radius $r_g = 10^{-4} \text{ m}$. In Figure 1 and 4 we consider cases where $\epsilon = .05$ or $\epsilon = .2$, that is the volume fraction of gas in the fluid is either 5% or 20%. The velocity profiles are presented for both the linear (Appendix A-1) and non-linear cases in dimensionless units.

For a small time interval the velocity gradients are very steep and linear theory is then generally in good agreement with the non-linear theory. This result is expected since, for t small, dissipative terms are more important than non-linear convective terms. As t increases, the wave propagates further into the liquid-gas mixture and the velocity gradients diminish, leading to a broadening of the shock-wave i.e., non-linear effects begin to dominate.

When $\epsilon \rightarrow 0$ the linear and non-linear solutions coalesce and shock fronts become infinitely steep; see Figures 1 and 2. Conversely, if ϵ increases then the two solutions diverge and the velocity gradients diminish.

The effects of altering the viscosity-like term θ are shown in Figures 5 and 6. Increasing θ (Figure 6) leads to a smearing of the velocity profile, which might be expected since θ contains terms related to momentum and thermal relaxation. If θ is reduced, on the other hand, then the velocity gradient becomes very large (Figure 5).

6. DISCUSSION

The propagation of pressure waves into gas bubbles suspended within a liquid is of some considerable interest. In this paper a mathematical model is developed which investigates the evolution of a shock wave propagating within a two-phase fluid, and proceeds with a description of how the compression wave forms and approaches the final steady state. The model developed is an extension of the previous linear model of van Leeuwen (1985) and is also a generalization of the work of Moran and Shen (1966), who considered the case of a shock wave propagating within a viscous heat-conducting gas. The linear and non-linear solutions of the flow that have been developed give a uniformly satisfactory description of the flow.

Important features of the present model are that the Navier-Stokes Equations for a liquid-gas mixture are solved using a perturbation expansion on the dependent variables and a dilation of the independent variables. This procedure extends the solution beyond the region where the linear solution breaks down. Decoupling the resultant partial differential equation leads to

Burger's equation for the velocity of the liquid. By using Laplace transform techniques this equation can be solved in closed form.

For small values of time the non-linear solution agrees well with the linear one, however for the far field the liquid velocity in the non-linear case runs ahead of the linear one.

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8. APPENDIX 1

The solution of the Navier-Stokes equations for unsteady flow to order $O(\epsilon)$ has been derived for a shock wave propagating into a liquid-gas mixture (van Leeuwen, 1984). The dimensionless solution for large times (i.e., the high frequency case) concomitant to (3.5b) can be written as:

$$u_l^{(1)}(x,t) = \frac{1}{2} \operatorname{erfc}\left(\frac{x - \lambda t}{\sqrt{2\theta t/\beta_e}}\right), \quad (\text{A-1})$$

and for the density, pressure and temperature of the gas as:

$$\rho_g^{(1)}(x,t) = p_g^{(1)}(x,t) = \frac{1}{2} \beta_e^{-1/2} (1-\beta_e) \operatorname{erfc}\left(\frac{x - \lambda t}{\sqrt{2\theta t/\beta_e}}\right), \quad (\text{A-2})$$

$$T_g^{(1)}(x,t) = 0. \quad (\text{A-3})$$

The velocity of the gas is simply:

$$u_g^{(1)}(x,t) = u_l^{(1)}(x,t). \quad (\text{A-4})$$

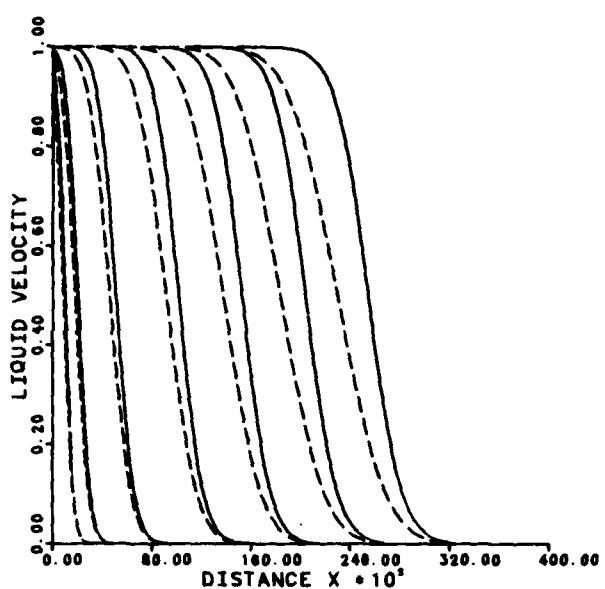


FIGURE 1
Liquid velocity profiles according to the Navier-Stokes equations; - - - linear solution, --- non-linear solution
 $\epsilon = .05$, $\beta_0 = .05$; $t = 200, 400, 1000, 2000, 3000, 4000, 5000$.

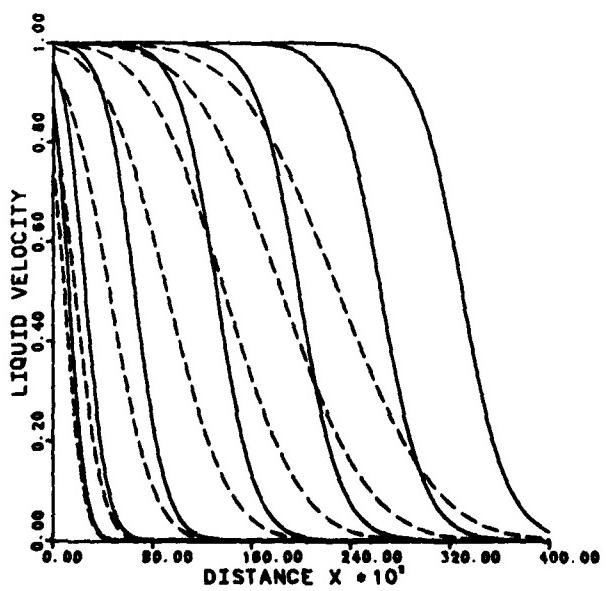


FIGURE 2
Liquid velocity profiles according to the Navier-Stokes equations; - - - linear solution, --- non-linear solution
 $\epsilon = .2$, $\beta_0 = .05$; $t = 200, 400, 1000, 2000, 3000, 4000, 5000$.

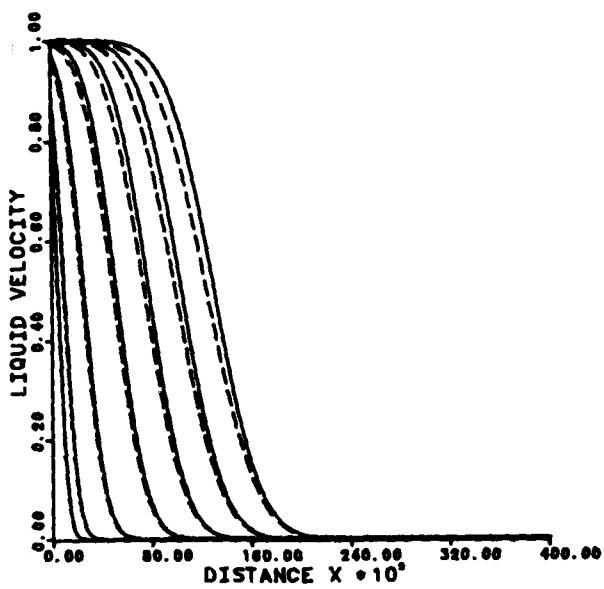


FIGURE 3 Liquid velocity profiles according to the Navier-Stokes equations; - - - linear solution, ---- non-linear solution
 $\epsilon = .05$, $\beta_0 = .2$; $t = 200, 400, 1000, 2000, 3000, 4000, 5000$.

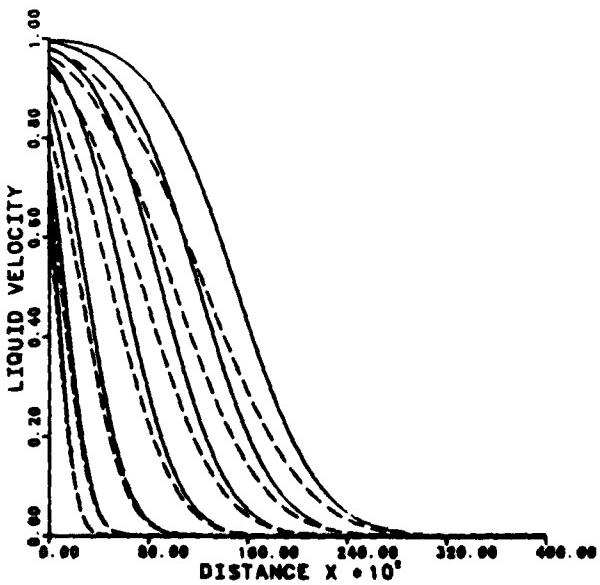


FIGURE 4 Liquid velocity profiles according to the Navier-Stokes equations; - - - linear solution, ---- non-linear solution
 $\epsilon = .2$, $\beta_0 = .2$; $t = 200, 400, 1000, 2000, 3000, 4000, 5000$.

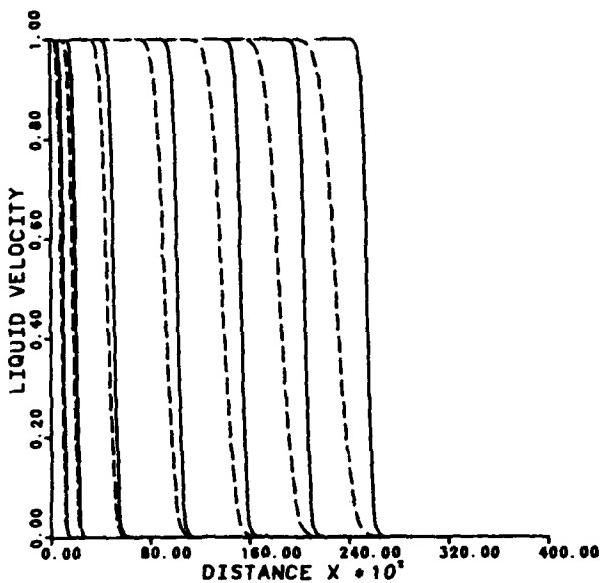


FIGURE 5 Liquid velocity profiles according to the Navier-Stokes equations; - - - linear solution, ---- non-linear solution
 $\epsilon = .05$, $\beta_0 = .05$, 0×10 ; $t = 200, 400, 1000, 2000, 3000, 4000, 5000$.

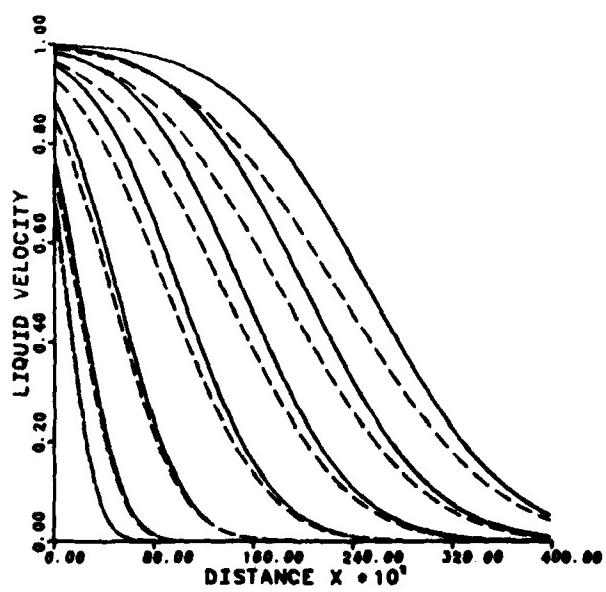


FIGURE 6 Liquid velocity profiles according to the Navier-Stokes equations; - - - linear solution, ---- non-linear solution
 $\epsilon = .05$, $\beta_0 = .05$, 0×10 ; $t = 200, 400, 1000, 2000, 3000, 4000, 5000$.

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